MIXED EFFECT AND SPATIAL CORRELATION MODELS FOR ANALYZING A REGIONAL SPATIAL DATASET

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Understanding the spatial variability of snow at many different scales is critically important for avalanche forecasting. This research uses new techniques to reanalyze an existing spatial dataset collected in southwest Montana's Bridger Range during the 1996-97 winter. Recent developments in statistical software have greatly increased the ease with which mixed effects and spatially correlated models can be run. Employing such advanced statistical procedures can lead to more beneficial use of available data sets and a more efficient use of limited financial funding. We reanalyzed the data using recently developed packages in SAS and the freely available software package R. The generalized linear model provides a suitable framework for categorical and / or dependent response variables. We analyzed the data using a fixed effect repeated measures model, a random effect clustered data model, and a spatially dependent fixed effects model using normally distributed continuous data. In addition, using the GLIMMIX procedure in SAS, we analyzed the data using a spatially dependent – random effects model with multinomial errors. This method allows for analysis of untransformed ordinal data, such as the data created when performing Rutschblock and stuffblock tests. Comparisons to the original analysis and suggestions for improving analytical efficiency are discussed. Our analyses help to provide a methodological context for future analyses of similar regional spatial data.

Keywords: mixed models, random effects, spatially dependent models, Bridger Mountains, stability tests.

1. INTRODUCTION

Understanding the spatial variability of snow at different scales is a critical component of avalanche forecasting. Motivations for investigating the effect of spatial variability are well documented (Schweizer et al, 2008). Few studies have collected concurrent data across large expanses, (> 40 km) thus enabling researchers to investigate only spatial variability without having to control for temporal changes. This paper revisits a data set collected during the winter of 1996-97 in southwest Montana's Bridger Mountains.

Complete univariate and linear regression analyses performed on this data were previously published by Birkeland (2001). The author correctly stresses a conservative approach using simple linear regression. However, recent advances in statistical software allow researchers with a moderate understanding of statistics to run mixed effects models and spatially dependent models. This can be accomplished with the widely available software program SAS (West et al, 2007; Littell et al, 2006) as well as the freely available program R (Pinheiro and Bates, 2002).

A detailed description of the model selection process is provided for one analysis. For ease of interpretation, results for each model are presented in the relevant sub-section. Inferences from these analyses will be compared to each other as well as to the original analyses in the discussion section. Linear mixed models, linear spatially dependent models and ordinal logistic with random spatial effects regression models are described.

2. SITE DESCRIPTION AND FIELD METHODS

The complete field methods, a full description of the study area, and a thorough discussion of the variables collected are described in Birkeland (2001). The data were collected during two days, February 6th, and April 2nd, 1997. The Bridger Range is classified in the intermountain avalanche climate zone, (Mock and Birkeland, 2000). The range consists of a single ridge approximately 40 km long and 10 km wide. The highest peaks climb 1400 m above the valley to reach an elevation of 2900 m.

Helicopters were used to transport six two-person crews to 13 locations along the main spine of the Bridger Mountain range during each of the two days of data collection. Not all

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response variables collected, nor environmental covariates collected are used in this paper.

For examples using continuous data, the Total Failure index (TFI) was used. To facilitate comparisons between locations with varying numbers of failure planes, Birkeland (2001) developed TFI using the following factors, stuffblock height (Birkeland and Johnson, 1999), failure depth, and number of failure planes. TFI decreases as avalanche hazard increases.

For examples using ordinal categorical data, rutschblock scores (Föhn, 1987) are used.

The three terrain variables used are elevation, slope angle, and radiation index (degrees from north).

3. MIXED EFFECTS MODELS

3.1 Overview of mixed effects models

When a regression contains both random and fixed effects, it is said to be a mixed effects model, or simply, a mixed model. Fixed effects are those with which most researchers are familiar. Any covariate that is assumed to have the same effect for all responses throughout the regression are considered fixed. Examples of fixed effects might be temperature, slope angle or aspect.

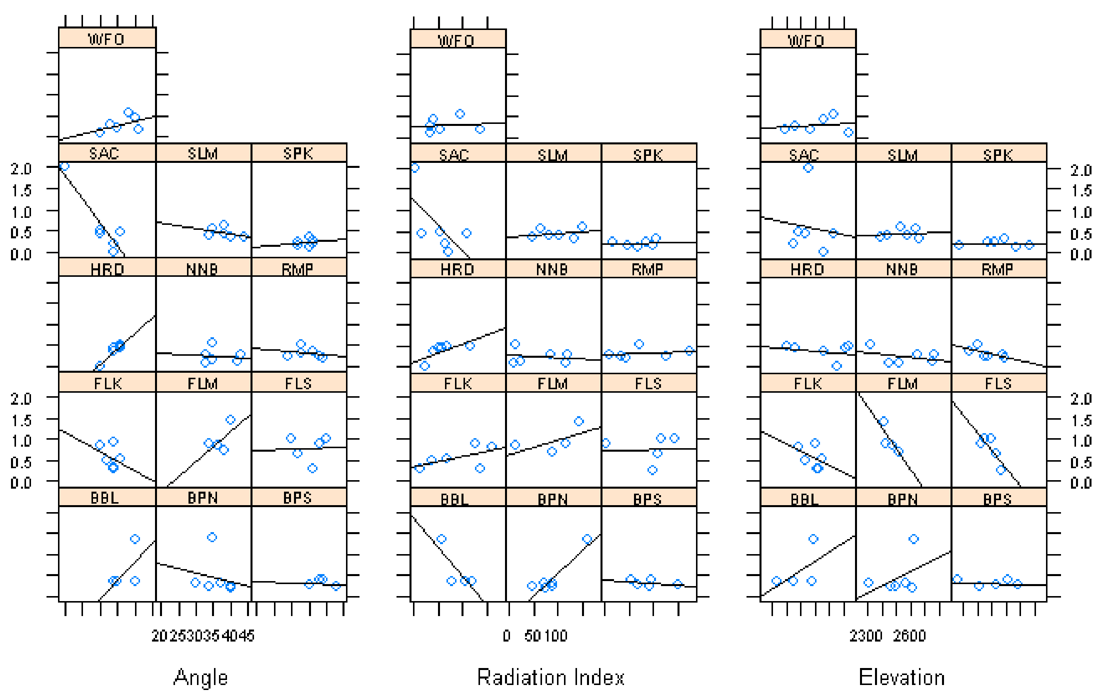
Random effects are those that are not of direct interest in a study. Examples of random effects in this study are observer effects, location effects, or possibly a day effect. They typically arise from logistical constraints placed on a study. By modeling random effects, more robust fixed effect coefficients and standard errors can be obtained. These coefficients are considered "population averaged", (West et al., 2007). If random effects are not properly modeled, then biased fixed effect coefficients are likely to be obtained.

The general matrix notation for a mixed model is;

$$\begin{aligned} &\mathrm{Y_i} = \mathrm{X_i}\beta + \mathrm{Z_i}u_i + \varepsilon_i \qquad (1) \\ &u_i \sim \mathrm{N}(0, \mathrm{D}) \\ &\varepsilon_i \sim \mathrm{N}(0, \mathrm{R}) \end{aligned}$$

where *X* is the data matrix, *β* is the vector of fixed-effect regression coefficients, *Z* is the observed values of covariates (can resemble *X* but typically has fewer columns), μ_i is the vector of random effects for the i^{th} subject, *D* is a variance-covariance matrix for the random effects and *R* is a variance-covariance matrix for the errors. By definition, random effects are random variables assumed to follow a normal distribution.

Fig. 1 Relationship between TFI and the three terrain variables at each location for the month of February



The motivation for using random effects can be best conveyed graphically. Figure 1 shows the relationship between the three terrain variables and TFI for the February data. The y-axis across all panels is TFI. The x-axes are elevation in meters, radiation index, and slope angle. Note that the slope and intercept for each variable vary between locations.

The simplest approach is to employ a random intercept model, one that allows for each location to have its own intercept. A slightly more complex model is one that has both a random intercept and random slope for each location.

3.2 <u>Two level clustered data model</u>

Initial inspection of the three terrain variables in relation to TFI during the February data collect indicates that a random slope and coefficient may be necessary (Figure 1).

The data described is a two level clustered data model wherein the first level is the snow stability response data, and the second level is the cluster of tests performed within each location. A top-down model selection approach (West et al., 2007) started with all three terrain variables of interest, the three two-way interactions, the one three-way interaction possible, and a random intercept.

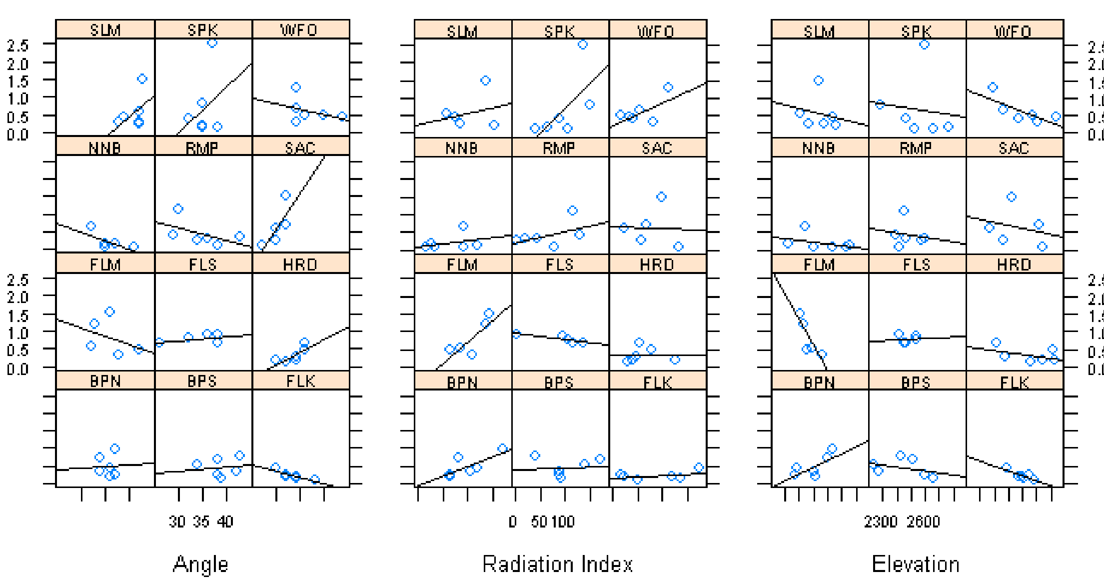
Model selection was carried out using an information theoretic approach, (Burnham and Anderson, 2001). The full model was fit with fixed effects only in order to determine which fixed effects were likely to remain in the model when the random effects were entered. Differences in Bayesian Information Criteria (BIC) values were used to determine the best fitting model. BIC values are based on maximum likelihood values (or a derivative such as REML), the number of parameters in the model, and the sample size of the data (Schwartz 1978). When comparing models with different fixed effects, maximum likelihood (ML) estimates were used. When comparing models with different random effects, restricted maximum likelihood (REML) estimates were used (West et al., 2007). Typically, the smaller the BIC value, the better the model. But this is only a guideline, not an absolute rule. Note that using REML instead of ML can drastically change the BIC values. Valid comparisons can be made only between models that have been fitted using the same method (i.e. REML or ML).

There is no one right way to proceed with model selection. For a complete discussion of the process see Burnham and Anderson (2001). For a detailed discussion of model selection with mixed models see West et al (2007)

3.3 <u>Model selection for mixed models</u>

A general description of the model selection process follows. A full model is fit using ML without any random effects. The p-values for individual effects are inspected and the least

Fig. 2 Relationship between TFI and the three terrain variables at each location for the April Data



significant effect is removed. This is repeated to determine the model with lowest BIC value. The only restriction of covariate removal was that if a two-way interaction remained in the model, each of the single covariates that comprise the interaction must be kept in the model. If a three-way interaction is left in the model, then all two-way interactions that comprise the three-way interaction must be left in the model. This is referred to as a hierarchical model. After a preliminary analysis of the fixed effects is completed, the best model is refit with REML and then compared to a model with a random intercept, and/or a random slope.

Once random effects are selected, all fixed effects must be re-inspected using ML. This is critical. Failure to revisit fixed effect selection can lead to spurious results and negates the purpose of using random effects.

3.4 Two- level mixed model for February data

A description of the model selection process for the February data follows.

Inspection of this model indicated the three-way interaction ($p = .83$) and two-way interaction between elevation and radiation ($p = .81$) were the least significant. The three-way interaction was removed. The following model, radiation* angle + angle* elevation + elevation* radiation resulted in a lower BIC value ($\Delta BIC = \sim 2$). This was repeated until a final suite of fixed effects were chosen. No clear suite of fixed effects emerged from the preliminary analysis. Therefore, the full suite of fixed effects, and a random intercept model were run.

A similar top-down model building approach was undertaken with no clear fixed effects showing as significant. Visual inspection of the data (Figure 2) indicates a model with both random slope and random intercept is needed. This and a model with only random slope were run using REML. The model with a random intercept and random slope resulted in the *highest* BIC value by a relatively small margin, ($\Delta BIC < 3$). This small margin coupled with the compelling visual evidence for both a random intercept and slope indicates the random slope and intercept model should be used.

Fixed effects were again inspected with a random slope model. The elevation only model, (BIC = 29.7) and the radiation only model (BIC = 29.9) had the lowest BIC values. In each model, the respective covariate was significant at the $\alpha = 0.1$ level (p-value = .07 and .09 respectively). The next two lowest models were the elevation + radiation (BIC = 31.2) and the elevation*radiation (BIC = 34.6). However, these later models did not yield any significant covariates at the $\alpha = 0.1$ level. The fact that that elevation and radiation fail to remain significant when included in the same model is further indication of the weak relationship both of these demonstrate for this data. A qq-norm test performed on the residuals of the final model determined that the errors were not normally distributed. The square root of TFI was calculated and the model was rerun. This did not change the two models selected as the best models, but resulting p-values changes for elevation ($p = 0.02$), and RI ($p = 0.06$)

A negative coefficient for elevation ($-6x10^{-4}$) suggests that lower elevations show a higher TFI. A positive coefficient for radiation ($1x10^{-3}$) suggests that larger RI values lead to higher TFI values. Thus, the more southerly slopes experience higher TFI values. The analysis suggests a weak link between terrain and snow stability, with more unstable conditions on northerly, high elevation slopes. These results are consistent with the conclusions of Birkeland (1997; 2001) for these data.

3.5 Two- level mixed model for April data

A similar analysis was performed on the April data (Figure 2). A random intercept and random slope model using the transformed response data (square root of TFI) yielded two significant covariates; elevation ($p = .008$) and radiation ($p = .006$). As with the February data, a negative coefficient for elevation ($-5.9x10^{-4}$) suggests higher TFI values at lower elevations and a positive coefficient for radiation ($1.9x10^{-3}$) suggests higher TFI values on more southerly slopes.

The stability on this day was more tightly linked to terrain, with more unstable conditions again on the higher elevation, northerly slopes. These analyses are again consistent with the results of Birkeland (2001) and the more detailed analyses of Birkeland (1997).

4. FIXED EFFECT REPEATED MEASURES MODELS

In a simple linear model, if the assumption of independent and identically distributed *(iid)* data holds, then $var(\varepsilon) = \sigma^2 I$ where I is the identity matrix. When the assumption of independence fails to hold, the variance can best be thought of as $var(\varepsilon) = \sigma^2\Omega$, where Ω is no longer a diagonal matrix and can be decomposed into Ω=DRD where D is the diagonal matrix of standard deviations and R is the correlation matrix. The main diagonal of a correlation matrix is naturally one.

The nlme package in R has the ability to fit non-identity matrix correlations. The gls (generalized least squares) function is used to call a variety of these correlations. This paper explores two. The first is a correlation matrix that simply calculates all off diagonal elements as the same value resulting in a fixed effect repeated measures model. The second is a spatially explicit correlation matrix that accounts for the distance between each pair of points. We will discuss the first here, and the second in the section relating to spatially dependent data (section 5).

4.1 Fixed effects model for February data

The best model for the February data using the fixed effects repeated measures analysis had the following covariates (regression coefficient shown in parentheses); elevation (-1.1×10^{-3}), radiation (-2.0×10^{-3}), angle (-6.0×10^{-2}), elevation* radiation (1.2×10^{-5}), and radiation* angle (6.1×10^{-4}).

All covariates and interactions in the chosen model are significant at the $\alpha=0.1$ level; elevation (p = .06), radiation (p = .004), angle (p = .0005), elevation* radiation (p = .08), and radiation* angle (p = .003).

Inferior model fit (ΔBIC > 11) is strong evidence that the fixed effects model does not explain the data as well as the mixed effects model. In addition, coefficients obtained from the fixed effects model will not be "population averaged", thus any inferences made to the entire Bridger Range will not be as robust.

4.2 Fixed effects model for April data

The model chosen using the fixed effects repeated measures analysis had the same fixed covariates as the model chosen using the mixed effect analysis. The coefficients for elevation (-5.9×10^{-4}) and radiation (1.9×10^{-3}) were essentially identical. Both have essentially the same BIC value (BIC = 19.55), indicating that both fit the data equally well.

5. FIXED EFFECTS SPATIAL LINEAR MODEL

With spatially dependent data sets, it is generally assumed (though not necessary) that the effect of distance is not dependent on direction (isotropy) or geographical location within the study area (stationarity) (Fortin and Dale, 2005). There are modeling approaches that can address egregious violations in each of these assumptions in a linear spatial model, but they will not be discussed here. Typically the patterns created from endogenous processes are referred to as spatial autocorrelation. The term spatial dependence is used here to define patterns generated from both exogenous and endogenous processes (Fortin and Dale, 2005).

Both the nlme package in R and the PROC MIXED procedure in SAS can adequately handle most spatial linear models for continuous data. Both allow for the following correlation structures; exponential, Gaussian, linear, and spherical. R can additionally utilize the rational quadratic correlation structure. SAS can run the Matern and power correlation structures as well.

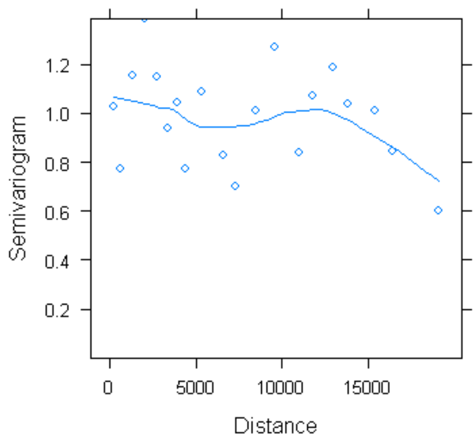
Methods that address spatially dependent data can be divided into two basic groups, characterization and adjustment, (Littell et al., 2006). The former typically is used to estimate covariance parameters and characterize the nature of correlation. The focus of this paper is on the latter, to remove the effects of spatial dependency in order to obtain more robust estimates of treatment effects. For an excellent introduction to adjustment models see chapter 10 of Hunsaker (2001).

The most important summary analysis when inspecting spatially dependent data (or data thought to be spatially dependent) is the semivariogram. The semivariogram can be defined in different ways. The x-axis is generally Euclidean distance between any two points. The y-axis can be one-half the variance of the difference between two observations (the default in SAS), or one minus the correlation between two observations (the default in R). The purpose of these two choices is identical. The intent is to estimate the following; the distance at which two observations can be thought of (or modeled as) independent (range), the variance that corresponds to the range (sill), and the distance at which two observations are considered to have a correlation of 1 (nugget). The shape of the function that best fits the semivariogram determines which spatial correlation model should be used.

The April data, square root of TFI, will be used as an example (Figure 3). A similar semivariogram is generated when using February data. The semivariogram indicates no spatial dependency.

In addition to fitting all observations at once, semivariograms for various classes of data were generated. It was thought that perhaps stability tests performed on similar aspects but large distances away from each other would be more correlated than tests performed near each other but on dissimilar aspects. This was investigated, but no spatial dependency was detected in TFI collected in April or February.

Figure 3. Variogram for April TFI data



The model was fit using the gls function in nlme. The fitted model was a much poorer fit than the models fit without spatial correlation, (ΔBIC > 20). This is expected, since the addition of parameters did not help reduce the variance. This was true for February data as well. This result is strong evidence that no spatial dependency is present in the TFI data.

6. ORDINAL LOGISTIC REGRESSION WITH RANDOM SPATIAL EFFECTS

Ordinal logistic regression (OLR) is an extension of logistic regression and has long been possible using standard statistical software. Recent software advances have allowed for ordinal logistic mixed models and ordinal logistic random spatial effect models.

In standard logistic models, the response data is binary, typically representing the presence or absence of an event or state. The response data in an OLR can be categorical in nature. This is especially important for avalanche data since these models will handle categorical data such as rutschblock, compression, and stuffblock test results. If the data are ordered in a sensible fashion, they are considered ordinal. All forms of logistic regression are part of a family of models called *generalized linear models*. The name arises from the fact that these models can be made linear through the use of a link function. The link function, $g(.)$ used for ordinal logistic regression extends the logit link function used in standard logistic regression. It is;

$$g\left(\mathrm{E}[Y]\right) = \log\left[\frac{\mathrm{P}(Y \le j \mid x)}{1 - \mathrm{P}(Y \le j \mid x)}\right] \qquad (2)$$

Random effects can be constructed by adding random effects to the linear predictor. The full ordinal logistic model with random spatial effects can be represented in matrix notation as;

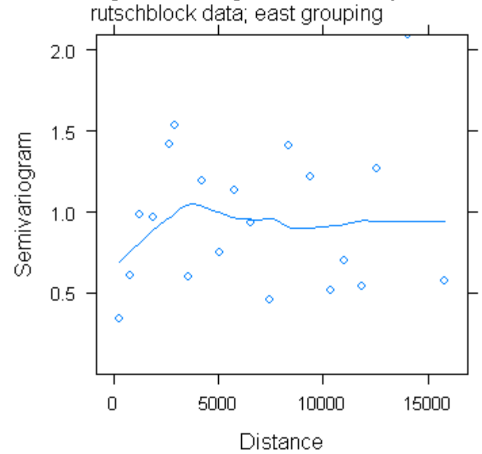
$$g\left(\mathrm{E}[Y_i \mid u]\right) = \mathrm{X}\beta + \varepsilon_i \qquad (3)$$
$$u_i \sim \mathrm{N}(0, \mathrm{D})$$

where Xβ represent the data matrix and coefficients as in equation 1. The random errors, u, are considered normal and centered on zero. The errors, ε_i, are multinomial errors and not normally distributed.

Our data include rutschblock results, which are naturally ordered from low stability (1) to high stability (7) (Föhn, 1989).

The GLIMMIX package in SAS will be used to demonstrate a spatially correlated ordinal logistic regression. No standard package in R allows for this model to be run.

Figure 4. Variogram for February rutschblock data; east grouping



An initial inspection of the Rutschblock data for February (Figure 4) indicated there might be spatial dependency within groups classified by aspect. The semivariogram does not suggest a strong relationship.

Solving generalized linear models with spatial dependency is computationally intensive. Good starting values are important. PROC GLMIMX provides a method for searching a user specified range for a good starting value. This is recommended if no clear starting value is apparent from the initial inspection of the semivariogram or previous analysis. Since there was no clear indication of the correlation approaching one as the distance approached zero in all three groups, a model with no nugget was used.

A spatial analysis using the February rutschblock data yielded three significant covariates. Radiation (p = .03), angle (p = .04) and the interaction of radiation and angle (p = .02) were significant at the α = .05 level. The intercepts for response category 2 through 6 are, respectively; -19.17, -17.77, -17.07, -15.96, and -14.12. Due to lack of convergence to a sensible range parameter, and the failure of the iterative procedure to converge on a positive definite G matrix, the 'nobounds' option was used in the GLIMIMIX statement. This allows the G matrix (which is comprised of all the D matrices shown in equations 1 and 2) to be negative. The resulting variance parameter (the sill) is -0.02, therefore the G matrix is not truly a variance-covariance matrix. Furthermore, valid inferences about the group specific intercepts can not be made, nor should the range parameter be directly interpreted. This does however allow for valid inferences to be made regarding the fixed parameters (West et al., 2007).

The ordinal logistic regression with random spatial correlation analysis for April yielded elevation (p = .04), radiation (p = .0007), and the interaction between elevation and radiation (p = .0007) as significant covariates. Elevation had a negative coefficient (-0.015), radiation had a negative coefficient as well (-1.094) and the interaction has a positive coefficient ($4.3x10^{-4}$). The intercepts for each response category, 1 through 6, are respectively; 31.51, 33.36, 33.77, 35.79, 37.12, and 38.77. The range parameter is 15,000 meters with a sill of 10.21 (SE = 12.15).

Comparisons using fit statistics to previously discussed models are inappropriate since they are not using the same data.

7. DISCUSSION

For February data, the mixed effects repeated measures analysis resulted in two models for consideration, each with one covariate. These two models, one with elevation only and one with only radiation, had very similar BIC values (ΔBIC < 1).

The fixed effect repeated measures analysis yielded a model with elevation, radiation, angle, elevation*angle and radiation*angle. While the three terrain variables and the associated interactions can all play a role in snow instability, this model has a significantly higher BIC score (ΔBIC > 11) suggesting this model is probably over fitting the existing data.

A fixed effect spatial linear model was fit in R and this yielded no improvement in fit for the continuous TFI data. This was expected since the semivariogram indicated spatial correlation was probably not present.

The ordinal logistic regression with random spatial correlation model was performed on the untransformed rutschblock data. Radiation, angle and the interaction between radiation and angle were found to be significant. This analysis confirms the importance of aspect in determining instability during the time the data were collected.

April mixed effects repeated measures analysis yielded elevation and radiation as significant covariates. The fixed effect repeated measures model yielded the same covariates. This model was fit using the fixed effects spatial linear model with no improvement in fit.

The ordinal logistic regression with random spatial correlation model was performed on the untransformed rutschblock data. Elevation, radiation, and the interaction between elevation and radiation were found to be significant covariates.

Birkeland (2001) imposed several rules for model selection. Among these were restrictions on significance level for ordered and non-normal data, as well as a requiring that no autocorrelation exist. While these restrictions were well suited to the analysis performed, this paper has presented several ways to appropriately relax those restrictions.

Mixed models are well suited for eliminating autocorrelation such that proper inferences can be made regarding the fixed effects of interest. In the case of the TFI data (transformed to TFIsqrt) collected in February, two valid models are presented here that were not presented in Birkeland (2001). These two univariate models, one with elevation and the other with radiation, did not meet Birkeland's criteria for achieving a significant p-value at the α = .05 level. However, they are presented here for comparison.

Using a spatially dependent ordinal logistic regression analysis, a valid model was obtained for the February rutschblock data. This model has three significant covariates, radiation, angle, and the interaction between the two (all significant at the α = .05 level). No valid model using only terrain covariates was presented by Birkeland (2001) in the original analysis.

When inspecting the April TFI data (transformed to TFIsqrt), no differences were found in the analysis of terrain variables. Both the results presented here and the original results (Birkeland, 2001) suggest that elevation and radiation (and no interaction) are the only significant terrain covariates.

When analyzing the rutschblock data using a spatially dependent ordinal logistic regression, the same covariates, elevation and radiation were shown to be significant. Birkeland added an additional covariate, UTM meters east that showed to be significant. That was not repeated since the spatial correlation was modeled using a random effect.

8. CONCLUSION

By modeling spatial correlation, either directly in the form of a correlation matrix, or indirectly in the form of repeated measures model, more rigorous inferences can be made on the covariates of interest. In this case those are terrain variables; elevation, slope angle, and radiation index.

Using an ordinal logistic model, with or without spatial correlation can increase efficiency when analyzing categorical data and lead to more robust inferences.

Example R code and SAS code can be found at *www.crazymountainresearch.org*.

Acknowledgements: We would like to thank the numerous individuals who helped collect these data. The U.S. National Science Foundation (Dissertation Improvement Grant No. SBR-962830) provided funding for the data collection.

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